# Quantization for Low-Rank Matrix Recovery

Eric Lybrand, Rayan Saab

# Overview

# Low Rank Matrix Recovery

Suppose $\mathcal{X} = \{X \in \mathbb{R}^{n_1 \times n_2} : \text{rank}(X) \leq k \ll n_1, n_2\}$

[1] www-bcf.usc.edu/

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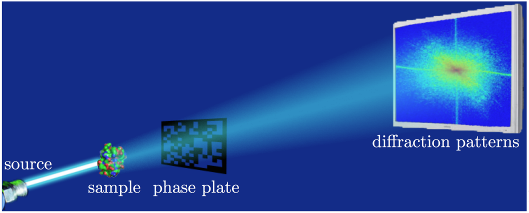
Quantum State Tomography, X-ray Crystallography

$$y_i = |\langle a_i, x\rangle|^2 = \langle a_i a_i^*, xx^*\rangle =: \mathcal{M}(xx^*)$$



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$$X^\sharp := \arg\min_Z \|Z\|_* \quad \text{subject to } \mathcal{M}(Z) = y,$$

$$\|Z\|_* = \sum_{j=1}^{r} \sigma_j(Z)$$

# Nuclear Norm Intuition

Low rank matrices have few singular values, i.e. vector of singular values is sparse

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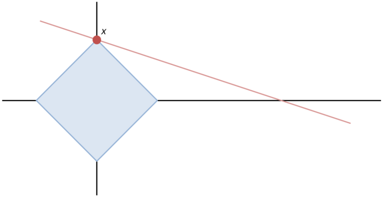
Use $\ell_1$ minimization

$$x^\sharp := \arg\min_Z \|x\|_1 \quad \text{subject to } \mathcal{M}(x) = y,$$

$$\|x\|_1 = \sum_{j=1}^{r} |x_j|$$

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# Random $\mathcal{M}$ Work Well

**Theorem (E. Candès, Y. Plan, '10)**

*Suppose $m \geq Ck \max\{n_1, n_2\}$, and let $\mathcal{M}(X) := \sum_{j=1}^{m} \langle A_i, X \rangle$ where $A_i$ are matrices with i.i.d. Gaussian entries. Then with high probability on the draw of $\mathcal{M}$ the following is true for all $X \in \mathbb{R}^{n_1 \times n_2}$ with $\mathrm{rank}(X) \leq k$:*
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More generally, linear maps which satisfy the matrix Restricted Isometry Property work well.

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Are the previous results robust to quantization error?

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In the simplest case, take

$$\mathcal{Q} : \mathbb{R} \to \{\pm 1\} \qquad \mathcal{D} : \{\pm 1\} \to \mathbb{R}$$

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**Problem:** It could be expensive to increase the number of bits used

# Oversampling

If the number of bits is fixed, try taking more measurements

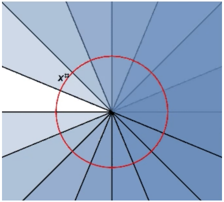
**Intuition:** Measurements $\text{sign}(\langle A_j, X\rangle)$ defines a half-space $X$ lies in.

Minimizing quantization error $\iff$ minimizing volumes

# The Shortcomings of MSQ

Volume of regions (i.e. reconstruction error) decay like $m^{-1}$

Vivek Goyal et al ('98): reconstruction error from MSQ quantized frame coefficients can't decay faster than $O(m^{-1})$.



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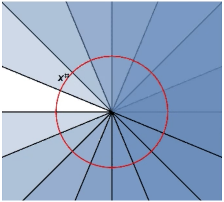

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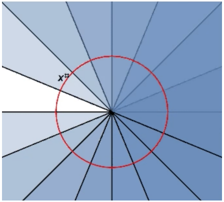
Alphabet resolution $\beta \implies \|y - q\|_2 \leq \sqrt{m}\beta \implies \|x - \hat{x}\|_2 \leq c_1\beta$ i.e. the error bound does not decrease with $m$.



# ΣΔ Quantization

Proposed by Inose & Yasuda, 1963 for quantizing bandlimited functions

Keeps track of $r$ previous quantization errors in a state variable $u$ to "shape" the quantized values

$$q_i = \mathcal{Q}(\rho_r(u_{i-1}, \dots, u_{i-r}, y_i, \dots, y_{i-r+1}))$$

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For example, could use equispaced grid where for some fixed $L > 0$ and resolution $\beta > 0$

$$\mathcal{A} := \{\pm(j - 1/2)\beta, \ j \in [L]\}.$$

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Trade off between bit complexity of alphabet and stability constant.

# A More General View of Noise Shaping

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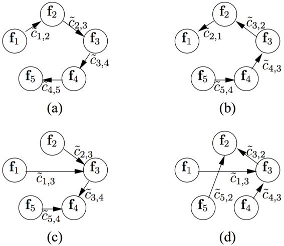
More general noise shaping could involving pushing quantization error for $\ell^{th}$ coefficient to the $\ell_k^{th}$ coefficient to compensate (Boufounos, 2006).

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# The Perks of Noise Shaping: Sparse Vectors

**Theorem (R. Saab, R. Wang, O. Yilmaz, 2015)**

*Let $A \in \mathbb{R}^{m\times N}$ be a Gaussian matrix with $m \geq C_1 k \log(eN/k)$. Then with high probability the following is true for any $k$-sparse $x \in \mathbb{R}^N$: let $q = \mathcal{Q}_{\Sigma\Delta}^{(r)}(Ax)$. The solution*

$$\hat{x} := \arg\min_z \|z\|_1 \quad s.t. \ \|D^{-r}(Az - q)\|_2 \leq \gamma(r)\sqrt{m}$$

*satisfies*

$$\|\hat{x} - x\|_2 \leq C_2 \beta \left(\frac{m}{\ell}\right)^{-r+1/2}.$$

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Result is robust to sparsity assumption.

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$$(X^\sharp, \nu^\sharp) := \arg\min_{(Z,\nu)} \|Z\|_* \quad s.t. \quad \|D^{-r}(\mathcal{M}(Z) + \nu - q)\|_2 \leq \gamma(r)\sqrt{m}$$
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*Then $X^\sharp$ satisfies*

$$\|X^\sharp - X\|_F \lesssim_r \left(\frac{m}{\ell}\right)^{-r+1/2} \beta + \frac{\sigma_k(X)_*}{\sqrt{k}} + \sqrt{\frac{m}{\ell}}\epsilon.$$

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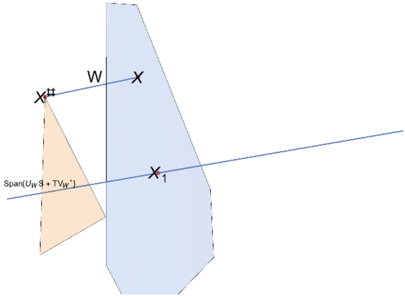

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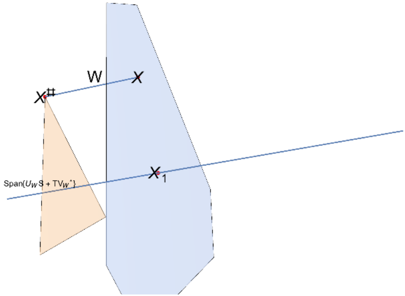
**New Goal:** Formulate a corresponding vector optimization problem where error between minimizer and truth is $\|W\|_F$.



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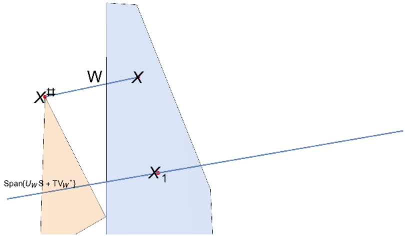

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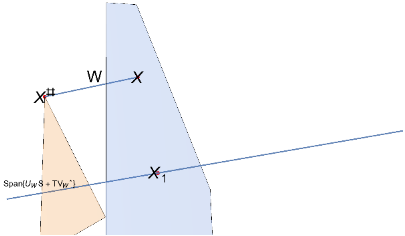

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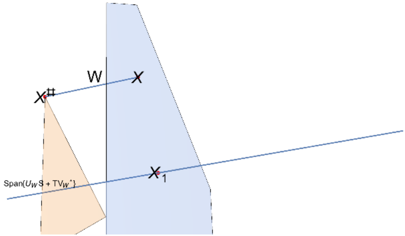
Show $\vec{\sigma}(W) - \vec{\sigma}(X)$ is feasible to the vector optimization problem with $A = M_{U_W, V_W}$ and $D^{-r} q = y$.





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Use a lemma from Oymak et al (2011) which buys us

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We use the chaining technique as proposed by Talagrand.

# Root Exponential Accuracy

## Corollary (E.L. and R. Saab, 2018)

*Let* $q = \mathcal{Q}^{(r)}_{\Sigma\Delta}(\mathcal{M}(X))$ *denote quantized noiseless measurements and* $X \in \mathbb{R}^{n_1 \times n_2}$ *with* $\mathrm{rank}(X) = k$. *Then there exist constants* $c, c_1, C_1, C_2 > 0$ *so that when*

$$\lambda := \frac{m}{\lceil ck \max(n_1, n_2) \rceil}$$

$$r := \left\lfloor \frac{\lambda}{2C_1 e} \right\rfloor^{1/2}$$

$$q := Q^{r}_{\Sigma\Delta}(\mathcal{M}(X)).$$

*the minimizer* $X^\sharp$ *satisfies* $\|X^\sharp - X\|_F \lesssim \beta e^{-c_1\sqrt{\lambda}}$.

# Exponential Accuracy with Random Encoding

**Corollary (E.L. and R. Saab, 2018)**

*Let $B : \mathbb{R}^m \to \mathbb{R}^L$ be a Bernoulli random matrix whose entries are $\pm 1$. Whenever $m \gtrsim_r L \gtrsim_r k \max(n_1, n_2)$ the following is true w.h.p. on the draw of $\mathcal{M}$ and $B$: the solution of*

$$(\hat{X}, \hat{\nu}) := \arg\min_{(Z,\nu)} \|Z\|_* \quad s.t. \quad \|BD^{-r}(\mathcal{M}(Z) + \nu - q)\|_2 \leq 3m\gamma(r)$$

$$\text{and} \quad \|\nu\|_2 \leq \epsilon\sqrt{m}.$$

*satisfies*

$$\|\hat{X} - X\|_F \lesssim \left(\frac{m}{L}\right)^{-r/2+3/4} \beta + \frac{\sigma_k(X)_*}{\sqrt{k}} + \sqrt{\frac{m}{L}}\varepsilon.$$

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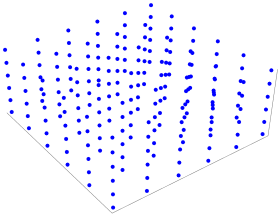
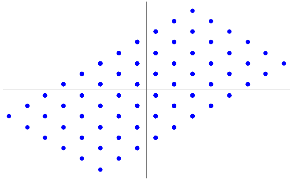
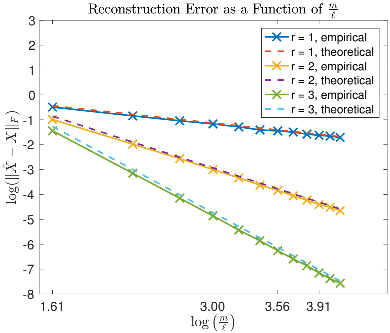
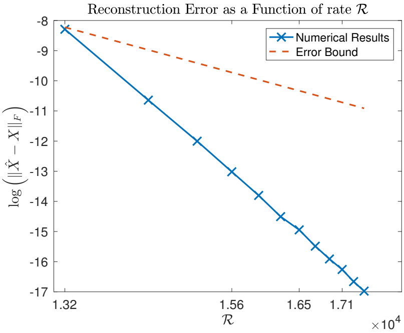
Random encoding "reduces complexity" of alphabet $\mathcal{A}$.





# Numerical Illustrations




Experimental DL: reconstruct rank 5, $20 \times 20$ Gaussian matrices from noiseless Gaussian measurements, averaged over 20 draws of true matrix.

# Future Directions

Taking sub-gaussian measurements is, in general, slow. Do the results hold for partial random circulant matrices, etc?

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Experiments show the exponent for noiseless encoding bound

$$\|\hat{X} - X\|_F \lesssim \left(\frac{m}{L}\right)^{-r/2+3/4}$$

is sub-optimal. Can we prove that it holds with the bound $\left(\frac{m}{L}\right)^{-r+3/4}$?

*Fin*

## Theorem (R. Saab, R. Wang, O. Yilmaz, 2015)

*Let $A \in \mathbb{R}^{m\times N}$, $P_\ell : \mathbb{R}^m \to \mathbb{R}^m$ the projection onto the first $\ell$ coordinates, and $V^*$ as in the singular value decomposition of $D^{-r}$. Suppose that $\frac{1}{\sqrt{\ell}} P_\ell V^* A$ has the vector-RIP of order $k$ and constant $\delta_k < 1/9$. Then any feasible $\hat{x}$ of*

$$(\hat{x}, \hat{\nu}) := \arg\min_{(z,\nu)} \|z\|_1 \quad s.t. \;\; \|D^{-r}(Az + \nu - q)\|_2 \leq \gamma(r)\sqrt{m}$$
$$\text{and} \;\; \|\nu\|_2 \leq \varepsilon$$

*with $\|\hat{x}\|_1 \leq \|x\|_1$ and $q$ satisfying $Ax + e - D^r u = q$ with $\|u\|_\infty \leq \gamma(r) < \infty$ and $\|e\|_2 \leq \varepsilon$ satisfies*

$$\|\hat{x} - x\|_2 \leq C\left( \left(\frac{m}{\ell}\right)^{-r+1/2} \beta + \frac{\sigma_k(x)_1}{\sqrt{k}} + \sqrt{\frac{m}{\ell}}\epsilon \right),$$

*where $\sigma_k(x)_1 = \text{argmin} \ldots \|x - z\|_1$*

# A Stronger RIP

## Definition

A linear map $\mathcal{M} : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^m$ satisfies the matrix RIP of order $k$ with constant $\delta_k$ if for any matrix $X$ with $\text{rank}(X) \leq k$

$$(1 - \delta_k)\|X\|_F^2 \leq \|\mathcal{M}(X)\|_2^2 \leq (1 + \delta_k)\|X\|_F^2$$

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## Lemma (S. Oymak, K. Mohan, M. Fazel, B. Hassibi, 2011)

*If $\mathcal{M}$ satisfies the matrix RIP of order $k$ with constant $\delta_k$, then for any unitary matrices $U, V$ the linear map $\mathcal{M}_{U,V}$ satisfies the vector RIP of order $k$ with constant $\delta_k$.*

# Proof Sketch

So it suffices to show that the linear map $\frac{1}{\sqrt{\ell}}P_\ell V^*\mathcal{M}$ satisfies the matrix RIP.

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**Goal:** Control

$$\mathbb{P}\left( \sup_X Z_X \geq t \right)$$

# The Generic Chaining

**Motivating Idea:** Suppose that $X$ were drawn from a finite set $T$. We could always union bound:

$$\mathbb{P}\left(\sup_{X\in T} Z_X \geq t\right) \leq \sum_{X\in T} \mathbb{P}\left(Z_X \geq t\right)$$

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Built off of an increment property: it is assumed that there exists a metric $d$ so that

$$\mathbb{P}\left(|Z_X - Z_Y| \geq t\right) \leq 2\exp\left(\frac{-t^2}{d^2(X,Y)}\right).$$

# The Generic Chaining

Successively approximate $Z_X$ by

$$Z_X = Z_X - Z_{\pi_1(X)} + Z_{\pi_1(X)} = Z_X - \sum_j Z_{\pi_j(X)} - Z_{\pi_{j-1}(X)}$$

where $\pi_j$ projects $T$ onto some finite subset $T_j \subset T$.
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Use the increment property on each of the residuals $Z_{\pi_j(X)} - Z_{\pi_{j-1}(X)}$ and union bound over the fibers of $\pi_j$.

# Results Using Chaining

Unsurprisingly, the geometry induced on $T$ by the metric $d$ will govern the tail bound

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The low dimensionality of the above set is what allows us to undersample and obtain the matrix RIP.